This experiment consists of determining the modal properties of a model structure. Many buildings in the practical world can be reasonably approximated by shear structures in which only one translational degree of freedom is taken into account for each floor level. Due to the simple characteristics of a shear structure, it is possible to determine its modal properties, such as the natural frequencies, damping, and mode shapes of all major modes from a well-performed modal test. A modal test is usually conducted to determine the modal properties of a structure and, if necessary, the analytical model of the structure can be modified accordingly for further simulation studies. Modal testing is especially useful for those structures with complex configurations for which it is difficult to get accurate results from analytical modeling.

In our treatment of this linear system, we explore the concept of the transfer function which describes how the structure responds to prescribed excitation. Since we will measure the input force (force of an impulse hammer blow, white noise from a shake table) and output (displacement recorded at specific locations), the transfer function, which characterizes the dynamic properties of the model, can be computed through simple time series or spectral analysis. Note that if the input function is more complex than a simple impulse, it can be separated into infinitesimal impulses. The total output (response) can be obtained by superposition of all the individual impulse responses. Once we know the transfer functions of a system, we can predict the response caused by any simple or complex source function.

The general procedure for performing a modal test is to select test points on the structure and collect transfer function data for each of these points. The model can be conveniently approximated as a lumped mass system at discrete points. These data points are then curve-fit to extract the modal properties of the test structure. A detailed discussion of application of modal tests and analysis to solve vibration problems can be found in Schmidtberg and Pal (1986) and the texts referenced below.

The technical objectives are to learn how to use the Dytran piezoelectric accelerometers, and to be able to record accurate results from impact hammer excitation. Another main objective is to become more familiar with the data acquisition system, its real-time data viewing and recording software, and Matlab for subsequent analysis. During the experiment, the results will be displayed in real time and also written to files for analysis after the experiment is over.

**Equipment**

- 5-story uniform shear beam model.
- Several Dytran Low Impedence Voltage Mode (LIVM) piezoelectric accelerometers.
- Data acquisition system. The system will take the analog signal received from the accelerometers and transform it into digital data. The digital signal will be viewed using the browser interface software with the Granite, and subsequent analysis will be done using Matlab.
- Calipers and ruler for measuring physical dimensions of the model.
- Scale to measure the weight of the structure.
- Shake table with several options for input signals including sinusoidal and white noise input.
- Notebook, pen and watch for recording the experimental setup and channel assignment into the laptop files.

To begin, we will discuss the test setup and the laptop configuration. We’ll discuss how to set parameters in the browser interface software. Then we will perform the impact hammer tests on the model. Next, we will impart a white noise signal into the base of the model. We will also do a free-vibration test on the model for an approximate calculation of the first mode damping. Last, we will disassemble parts of the model to obtain the measurements necessary to calculate the mass and stiffness matrices of the model.
**Report**

Supply a report with all the sections as discussed in the syllabus and class web site. Sketch the arrangement of the experimental apparatus and all experimental conditions. From the free oscillation test, determine the approximate damping of the first mode from the time series (amplitude decay method). From the impact hammer tests, determine the five modal frequencies for the model using computed transfer functions, approximate values of damping for each mode (half-power method) and mode shape profiles. Using white noise input signal to the base of the structure, determine the natural frequencies and modal damping values. Compare the ratio of these frequencies to the ratio of an idealized 5-story shear beam model. Also compare the mode shape profiles. At the end of the experiment, we will disassemble the top floor of the model and record the physical properties of the model using calipers and a scale. You need this information to compute the mass and stiffness matrices and to calculate the mode shapes and natural frequencies analytically.

**References**


Class Material, AM/CE 151: Dynamics and Vibrations

Class Material, CE 181: Engineering Seismology
**Theoretical background**

Consider a structure which is modeled by a multi-degree-of-freedom (MDOF) system. Its equation of motion can be written as

\[ m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \]  

(1)

where \( m \), \( c \), and \( k \) are the mass, damping, and stiffness matrices associated with the system. Note that all three of the above matrices are symmetric, reflecting structural reciprocity. Taking Fourier transforms of Eq. (1), we obtain

\[ (-\omega^2 m + i\omega c + k)X(\omega) = F(\omega) \]  

(2)

where \( X(\omega) \) and \( F(\omega) \) are the Fourier transforms of the displacement response \( x(t) \) and the excitation \( f(t) \), respectively. The matrix of transfer functions (or frequency response functions), \( G(\omega) \), is defined through the relation

\[ X(\omega) = G(\omega)F(\omega). \]  

(3)

Thus, by Eqs. (2) and (3), we get

\[ [G(\omega)]^{-1} = -\omega^2 m + i\omega c + k \]  

(4)

Now convert this to an eigenvector/eigenvalue problem. Assume that \( A \) is the modal matrix of the N-DOF system. I.e.,

\[ [A] = [\phi_1, \phi_2, ..., \phi_N] \]

where \( \phi_i \) is the \( i^{th} \) mode shape. Then, using the orthogonality of mode shapes with respect to \( m \), \( c \) and \( k \), and assuming that the system has proportional (viscous) damping,

\[ c = \alpha m + \beta k, \]

we can derive from Eq. (4)

\[ A^T[G(\omega)]^{-1}A = A^T(-\omega^2 m + i\omega c + k)A \]

\[
\begin{pmatrix}
-\omega^2 M_{11} + i\omega C_{11} + K_{11} & 0 & 0 & 0 & 0 \\
0 & . & 0 & 0 & 0 \\
0 & 0 & . & 0 & 0 \\
0 & 0 & 0 & . & 0 \\
0 & 0 & 0 & 0 & -\omega^2 M_{NN} + i\omega C_{NN} + K_{NN}
\end{pmatrix}
\]

(5)

where \( M_r, C_r, K_r \) (where \( r=1, ..., N \)) are the modal mass, damping, and stiffness, respectively, associated with the system. Hence,
\[
G(\omega) = \begin{pmatrix}
\frac{1}{-\omega^2 M_{11} + i \omega C_{11} + K_{11}} & 0 & 0 & 0 \\
0 & \frac{1}{-\omega^2 M_{22} + i \omega C_{22} + K_{22}} & 0 & 0 \\
0 & 0 & \frac{1}{-\omega^2 M_{33} + i \omega C_{33} + K_{33}} & 0 \\
0 & 0 & 0 & \frac{1}{-\omega^2 M_{44} + i \omega C_{44} + K_{44}}
\end{pmatrix} A^T
\]

or

\[
G_{km}(\omega) = \sum_{r=1}^{N} \frac{\phi_r^k \phi_r^m}{-\omega^2 M_{rr} + i \omega C_{rr} + K_{rr}} = \sum_{r=1}^{N} M_{rr} \left( -\omega^2 + 2i \zeta_r \omega + \omega^2 \right)
\]

where \(\phi_r^k\) is the \(k^{th}\) component of the \(r^{th}\) mode shape, \(\zeta_r\) is the modal damping and \(\omega_r\) is the frequency of the \(r^{th}\) mode. Note that the transfer function \(G_{km}(\omega)\) relates the response at DOF \(k\) to the simple harmonic excitation at DOF \(m\). Recall that \(\omega_r^2 = K_r / M_r\) and \(2\zeta_r \omega = C_r / M_r\).

From Eq. (6), it is easy to see that the transfer function matrix is symmetric; i.e.,

\[
G_{km}(\omega) = G_{mk}(\omega)
\]

This reciprocal property allows us to obtain the mode shapes of any MDOF system by measuring the response at only one DOF with excitation applied at different locations.

The real and imaginary parts, and the amplitude of the transfer function can be directly derived from Eq. (6)

\[
\text{Real}(G_{km}(\omega)) = \sum_{r=1}^{N} \frac{\phi_r^k \phi_r^m}{-\omega^2 M_{rr} + i \omega C_{rr} + K_{rr}} \left( \omega_r^2 - \omega^2 \right)
\]

\[
\text{Im}(G_{km}(\omega)) = -\sum_{r=1}^{N} \frac{\phi_r^k \phi_r^m}{-\omega^2 M_{rr} + i \omega C_{rr} + K_{rr}} \frac{2\zeta_r \omega_r \omega}{\left( \omega_r^2 - \omega^2 \right)^2 + (2\zeta_r \omega_r \omega)^2}
\]

\[
|G_{km}(\omega)| = \sum_{r=1}^{N} \frac{\phi_r^k \phi_r^m}{M_{rr} \sqrt{\left( \omega_r^2 - \omega^2 \right)^2 + (2\zeta_r \omega_r \omega)^2}}
\]

Considering only the contribution of a particular mode \(r\) at frequency \(\omega = \omega_r\), we see that the real part is zero and the transfer function is a purely imaginary quantity.

\[
G_{km}(\omega_r) = -\frac{\phi_r^k \phi_r^m}{M_{rr} \left( 2\zeta_r \omega_r^2 \right)} i.
\]

Therefore, by measuring the imaginary parts of \(G_{km}(\omega_r)\) for \(m=1, \ldots, N\), we can obtain the complete mode shape \(\phi_r\) for particular mode \(r\), by computing a ratio of transfer functions.
\[
\frac{G_{km}(\omega_r)}{G_{mm}(\omega_r)} = \frac{\phi_r^k \phi_r^m}{\phi_r^m \phi_r^m} = \frac{\phi_r^k}{\phi_r^m}
\]

if all the peak values of the transfer functions at \(\omega_r\), can be clearly identified. Note that in using Eq. (9) for determination of mode shapes, we retain the phase information (positive or negative deflections) of the mode shapes. Recall that if we simply look at the amplitude spectrum, we are not getting phase information.

It should be mentioned that in using Eq. (9) we neglect the contributions from modes other than \(r\). This is acceptable when the modes of the structure are well-separated, as is the case for a simple shear structure. It allows us to neglect contributions from all eigenfunctions except the one corresponding to the modal frequency we’re interested in. Usually curve fitting to Eq. (9) is needed to obtain a more accurate result for mode shapes and modal damping values.

Eq. (10) gives the \(k^{th}\) component of the \(r^{th}\) mode shape normalized with respect to the \(m^{th}\) component of the same mode shape. \(G_{mm}(\omega_r)\) and \(G_{km}(\omega_r)\) can be measured by two ways. The first is to provide an excitation at the \(m^{th}\) degree of freedom and measure the response at that degree of freedom and then provide the excitation at the \(k^{th}\) degree of freedom and again measure the response at the \(m^{th}\) degree of freedom. The other option is to provide the excitation at the \(m^{th}\) degree of freedom and measure the response at both the \(m^{th}\) and \(k^{th}\) degrees of freedom.

The assumption of neglecting the contributions from neighboring modes also needs to be implemented when determining the modal damping coefficient using the half-power method. This assumption is valid for lightly damped shear buildings where the slopes of the transfer function curve are usually steep. Using this assumption, the amplitude of the transfer function in the neighborhood of an eigenfrequency is given by

\[
|G_{km}(\omega)|_{\text{near } \omega_r} = \frac{\phi_r^k \phi_r^m}{M_{rr}} \frac{1}{\sqrt{(\omega_r^2 - \omega^2)^2 + (2\zeta_r \omega_r \omega)^2}}. \tag{11}
\]

Note that \(\frac{\phi_r^k \phi_r^m}{M_{rr}}\) is a constant and can be considered a multiplication factor. Thus, the above transfer function actually yields an estimate for the expression

\[
\frac{1}{\sqrt{(\omega_r^2 - \omega^2)^2 + (2\zeta_r \omega_r \omega)^2}} \tag{12}
\]

which corresponds to the displacement amplitude response in the frequency domain. According to the half-power method, the modal damping coefficient can be determined from the above expression. The maximum amplitudes are determined for frequencies \(f_1\) and \(f_2\), where the amplitudes of the response are equal to \(1/\sqrt{2}\) times the maximum amplitude. The damping ratio is then given as

\[
\zeta \approx \frac{f_2 - f_1}{f_2 + f_1}. \tag{13}
\]

It is common to equate the denominator in Eq. (13) to \(2f_n\) which is the natural frequency of the mode. However, Eq. (13) is preferable, especially in the case where the amplification curve is not known with good detail (e.g., if there are large frequency damping steps). Note that the half-power method is based on the response of a SDOF system for which it yields the exact damping value. It can be used for a MDOF system as long as the modal frequencies are distinct so that there is no strong correlation (i.e., coupling) between the modes. In the range of frequencies near each mode, the MDOF system can be approximated
as a SDOF system and the above equation is valid. Note that the half-power method uses displacement as the response parameter, so the transfer function with respect to the displacement should be calculated.

**Shear Model Structure: Analytical Approach**

If all the masses and stiffness coefficients can be calculated, the eigenproblem can be solved for theoretical eigenfunctions and eigenfrequencies of the model structure. If we assume that the model structure behaves as an ideal shear structure, then the mass matrix can be determined by measuring all the masses of the model floors, and the stiffness matrix components can be calculated. Note that the stiffness matrix is nondiagonal, implying coupling of equations and the need to solve equations simultaneously.

Following Chopra (2002), the lateral stiffness $k_j$ of the $j^{th}$ story relates the story shear $V_j$ to the story deformation or drift, $\Delta_j = u_j - u_{j-1}$, by $V_j = k_j \Delta_j$ assuming linear behavior. The story stiffness is the sum of the lateral stiffnesses of all columns in the story. For a story of height $h$ and a column with Young’s elastic modulus $E$ and second moment of area $I$, the lateral stiffness of a column with clamped ends, implied by the shear-building idealization, is $12E I / h^3$. Thus the story stiffness is

$$k_j = \sum_{\text{columns}} \frac{12EI_c}{h^3}.$$  

When the mass and stiffness matrices are populated with measured values of our model structure, the equations of motion can be solved simultaneously for frequencies of vibration and modal shapes.

First, describe displacement of structure in one of its modes $n$

$$u(t) = q_n(t) \phi_n$$

where $\phi_n$ is the deflected mode shape and

$$u(t) = \phi_n(A_n \cos \omega_n t + B_n \sin \omega_n t).$$

Substituting this form of $u(t)$ into the equation of motion for undamped system gives the matrix eigenvalue problem

$$k \phi_n = \omega^2 m \phi_n$$

where $m$ and $k$ are known, but scalar $\omega_n$ and vector $\phi_n$ are not.

These values can be obtained by solving the eigenvalue problem represented by

$$\left\{ k - \omega^2 m \right\} \phi = 0.$$  

Note that in this ideal shear building, we have assumed that the beams and floor systems are rigid (infinitely stiff) in flexure. However, the assumption of shear deformation is convenient for understanding structural response, and in many real-world cases is a close approximation to actual building response. The natural frequencies and mode shapes of a uniform shear structure with $N$ DOF can be found as

$$\omega_k \approx 2 \sin \left( \frac{(2k-1)\pi}{2(2N+1)} \right)$$
where $C_k$ is an arbitrary normalizing constant. It is interesting to note that as $N \to \infty$, $\omega_k \sim 2k - 1$. For the case of a three-DOF system, we have

$$\omega_1 : \omega_2 : \omega_3 \doteq 1 : 2.8 : 4.1$$

and

$$\begin{pmatrix} 0.434 \\ 0.782 \\ 0.975 \end{pmatrix} \text{, } \begin{pmatrix} 0.975 \\ 0.434 \\ -0.782 \end{pmatrix} \text{, and } \begin{pmatrix} 0.782 \\ -0.975 \\ 0.434 \end{pmatrix}.$$